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Global stabilizability of uncertain systems with time-varying delays via dynamic observer-based output feedback

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Abstract

In this paper, the dynamic observer-based output feedback control for uncertain time-delay systems with time-varying delays is investigated. A delay-independent dynamic observer-based output feedback control is proposed such that the feedback-controlled system with time-varying delays is globally asymptotically stable if some mild conditions are met. An upper bound of arbitrary time-varying delays without destroying stability is also given such that the asymptotic stability is preserved. A numerical example is given to illustrate the use of our main results.

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1. Introduction

Time delay is often encountered in various areas, such as chemical engineering systems, AIDS epidemic, the aircraft stabilization, the ship stabilization, the manual control, the turbojet engine, the nuclear reactor, the microwave oscillator, the rolling mill, and systems with lossless transmission lines. It is frequently a source of instability and a source of generation of oscillation in many control systems [5]. The feedback control of time-delay systems, with or without uncertainties, has been extensively studied in recent years. In particular, the global asymptotic stabilization

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for time-delay systems, with or without uncertainties, constitutes an important area for practical control design. In the past, there have been a number of interesting developments in seeking stabilizability criteria for time-delay systems, but most were restricted to delay-dependent stabilizability criteria for systems with constant delays, or delay-independent criteria for systems with time-varying delays, or delay-dependent criteria for systems with time-varying delays $h_i(t)$ but subjected to the assumption $\dot{h}_i(t) < 1$ [1,6–12]. It is well known that a static (or dynamic) output controller, which uses only the output for feedback, is more practical to deal with uncertain systems [3,4]. In particular, the observer-based output feedback controller, which is a dynamic output feedback controller, can on-line estimate the system states. It is the purpose of this paper to propose a dynamic observer-based output feedback control such that not only the feedback-controlled system with time-varying delays is globally asymptotically stable but also the estimated state trajectories asymptotically track the true state trajectories of the feedback-controlled system with time-varying delays. In addition, it will be shown that even if (A_0, B, C) is not jointly stabilizable and detectable, the system $\dot{x}(t) = A_0x(t) + \sum_{i=1}^{p} A_i x(t - h_i(t)) + Bu(t)$, $y(t) = Cx(t)$, may still be globally asymptotically stabilized by a dynamic observer-based output feedback. An upper bound of arbitrary time-varying delays without destroying stability is also given such that the asymptotic stability is preserved.

This paper is organized as follows. In Section 2, the problem formulation is presented. A delay-independent dynamic observer-based feedback control is proposed such that the feedback-controlled system with time-varying delays is globally asymptotically stable if some mild conditions are met. Finally, a numerical example is provided to illustrate the main results in Section 3.

2. Problem formulation and main results

For convenience, we define some notation that will be used throughout this paper as follows:

$\mathbb{R}$:= is the set of all real numbers,
$\mathbb{R}^n$:= the n-dimensional real space,
$\mathbb{R}^{m\times n}$:= the set of all real m by n matrices,
A^{T} := the transpose of the matrix A,
I := the unit matrix,
$\|A\|$:= the induced Euclidean norm of the matrix A,
H_c := the set of all matrices whose eigenvalues have negative real parts,
$\lambda_{\max}(Q)$ (res. $\lambda_{\min}(Q)$) := the maximum (res. minimum) eigenvalue of the symmetric matrix Q,
$Q > 0$:= the symmetric matrix Q is positive definite,
$C(t_1, H, \mathbb{R}^n) := \{\phi : [t_1 - H, t_1] \to \mathbb{R}^n \mid \phi \text{ is continuous}\}$,
$\underline{p} := \{1, 2, \ldots, p\}$,
$\overline{p} := \{0, 1, 2, \ldots, p\}$.

In this paper, we consider the uncertain time-delay system with multiple time-varying delays described as

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{p} A_i x(t - h_i(t)) + \Delta f(t, x(t), x(t - h_1(t)), \ldots, x(t - h_p(t))) + Bu(t), \quad t \geqslant 0, \tag{1a}$$

$$y(t) = (C + \Delta C(t))x(t) \quad \forall t \geqslant 0, \tag{1b}$$

$$x(t) = \theta(t), \quad t \in [-H, 0], \tag{1c}$$

where $x \in \mathbb{R}^n$ is the state vector, $y \in \mathbb{R}^r$ is the output vector, $u \in \mathbb{R}^d$ is the input vector, $A_i \in \mathbb{R}^{n \times n}$, $\forall i \in \bar{p}$, $C \in \mathbb{R}^{r \times n}$, $h_i(t)$'s, $\forall i \in \underline{p}$, are arbitrary delay arguments with $0 \leqslant h_i(t) \leqslant H$ for some constant H, and $\theta(t)$ is a given continuous vector-valued initial function. In addition, we assume that ΔC and Δf, the uncertain terms, are smooth vector-valued functions to guarantee the existence of the solution for (1).

The following assumption is made on the system (1) throughout this paper.

(A1). There exist non-negative constants a_i's $\forall i \in \bar{p}$ and c, such that, for all arguments,

$$\|\Delta f(t, z_0, z_1, \ldots, z_p)\| \leqslant \sum_{i=0}^{p} a_i \cdot \|z_i\|, \qquad \|\Delta C(t)\| \leqslant c.$$

Now we present our first main result for the global asymptotic stabilizability of the system (1).

Theorem 2.1. *The system* (1) *satisfying* (A1) *is globally asymptotically stabilizable by a dynamic observer-based output feedback provided that there exist matrices* $Q > 0$, $K \in \mathbb{R}^{n \times d}$, $L \in \mathbb{R}^{n \times r}$, *and two sets* $L_1, L_2 \subseteq \underline{p}$ *such that*

(i) $(\bar{A}, B)$ *is stabilizable with* $\bar{A} := A_0 + \sum_{i \in L_1} A_i$;

(ii) $(\tilde{A}, C)$ *is detectable with* $\tilde{A} := A_0 + \sum_{i \in L_2} A_i$;

(iii) $$\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)} \cdot \sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}} > 2\sum_{i=0}^{p} a_i + \sum_{i \in \underline{p} \backslash L_1} \|A_i\| + \sum_{i \in \underline{p} \backslash L_2} \|A_i\| + c\|L\| + H\left[\left(\sum_{i \in L_1} \|A_i\|\right) \cdot \left(\sqrt{2}\|BK\| + \sum_{i=0}^{p} \left(\|A_i\| + a_i\right)\right) + \left(\sum_{i \in L_2} \|A_i\|\right) \cdot \left(\sqrt{2}\|BK\| + \sum_{i=0}^{p} \left(\|A_i\| + a_i\right)\right)\right],$$

where $P > 0$ *is the unique solution to the following Lyapunov equation*:

$$\begin{bmatrix} \bar{A} + BK & -BK \\ 0 & \tilde{A} - LC \end{bmatrix}^{\mathrm{T}} P + P \begin{bmatrix} \bar{A} + BK & -BK \\ 0 & \tilde{A} - LC \end{bmatrix} = -Q \tag{2}$$

with $\bar{A} + BK \in H_c$ *and* $\tilde{A} - LC \in H_c$. *In this case, a suitable dynamic observer-based output feedback is given by*

$$u(t) = K\hat{x}(t), \tag{3}$$

$$\dot{\hat{x}}(t) = (A_0 - LC + BK)\hat{x}(t) + \sum_{i \in L_2} A_i \hat{x}(t) + Ly(t). \tag{4}$$

Proof. Without loss of generality, we may let $L_1 = \{1, 2, \ldots, m_1\} \subseteq \underline{p}$ and $L_2 = \{1, 2, \ldots, m_2\} \subseteq \underline{p}$. Define

$$e(t) = x(t) - \hat{x}(t). \tag{5}$$

Then, from (1), (3), and (5), we have

$$\begin{aligned} \dot{x}(t) = {} & A_0 x(t) + \sum_{i=1}^{m_1} A_i x(t - h_i(t)) + \sum_{i=m_1+1}^{p} A_i x(t - h_i(t)) \\ & + \Delta f + BK\hat{x}(t) \\ = {} & A_0 x(t) + \sum_{i=1}^{m_1} A_i x(t) - \sum_{i=1}^{m_1} A_i \int_{t-h_i(t)}^{t} \dot{x}(s)\, \mathrm{d}s \\ & + \sum_{i=m_1+1}^{p} A_i x(t - h_i(t)) + \Delta f + BK(x(t) - e(t)) \\ = {} & (\bar{A} + BK)x(t) \\ & - \sum_{i=1}^{m_1} A_i \int_{t-h_i(t)}^{t} \left[\sum_{j=0}^{p} A_j x(s - h_j(s)) + \Delta f + Bu \right] \mathrm{d}s \\ & + \sum_{i=m_1+1}^{p} A_i x(t - h_i(t)) + \Delta f - BKe(t), \quad t \geqslant 0, \end{aligned} \tag{6a}$$

$$x(t) = \theta(t), \quad t \in [-H, 0], \tag{6b}$$

with $h_0(t) := 0$. Furthermore, from (5), (6a), and (4), we have

$$\begin{aligned} \dot{e}(t) = \dot{x}(t) - \dot{\hat{x}}(t) = {} & A_0 e(t) + \Delta f - LCe(t) + L\Delta Cx(t) \\ & + \sum_{i=1}^{m_2} A_i \big[x(t - h_i(t)) - \hat{x}(t) \big] + \sum_{i=m_2+1}^{p} A_i x(t - h_i(t)) \end{aligned}$$

$$
\begin{aligned}
&= (A_0 - LC)e(t) + \Delta f + L\Delta C x(t) - \sum_{i=1}^{m_2} A_i[x(t) - x(t - h_i(t))] \\
&\quad + \sum_{i=1}^{m_2} A_i e(t) + \sum_{i=m_2+1}^{p} A_i x(t - h_i(t)) \\
&= (\tilde{A} - LC)e(t) + \Delta f + L\Delta C x(t) - \sum_{i=1}^{m_2} A_i[x(t) - x(t - h_i(t))] \\
&\quad + \sum_{i=m_2+1}^{p} A_i x(t - h_i(t)) \\
&= (\tilde{A} - LC)e(t) + \Delta f + L\Delta C x(t) - \left(\sum_{i=1}^{m_2} A_i \int_{t-h_i(t)}^{t} \dot{x}(s)\,\mathrm{d}s\right) \\
&\quad + \sum_{i=m_2+1}^{p} A_i x(t - h_i(t)) \\
&= (\tilde{A} - LC)e(t) + \Delta f + L\Delta C x(t) + \sum_{i=m_2+1}^{p} A_i x(t - h_i(t)) \\
&\quad - \left\{ \sum_{i=1}^{m_2} A_i \int_{t-h_i(t)}^{t} \left[A_0 x(s) + \left(\sum_{j=1}^{p} A_j x(s - h_j(s)) \right) \right.\right. \\
&\qquad \left.\left. + \Delta f + Bu(s) \right] \mathrm{d}s \right\}. \qquad (6c)
\end{aligned}
$$

Define the dynamic system

$$
\begin{aligned}
\dot{z}(t) &:= \begin{bmatrix} \dot{y}_1(t) \\ \dot{e}(t) \end{bmatrix} := F(t, z_t) \\
&:= \begin{bmatrix} F_1(t, z_t) \\ F_2(t, z_t) \end{bmatrix} = \begin{bmatrix} \bar{A} + BK & -BK \\ 0 & \tilde{A} - LC \end{bmatrix} \begin{bmatrix} y_1(t) \\ e(t) \end{bmatrix} \\
&\quad + \begin{bmatrix} \Delta g_1 \\ \Delta g_2 \end{bmatrix} \quad \forall t \geqslant 0, \qquad (7a)
\end{aligned}
$$

$$y_1(t) = \theta(t), \quad t \in [-H, 0], \qquad (7b)$$

$$y_1(t) = 0, \quad t \in [-2H, -H], \qquad (7c)$$

where

$$\begin{aligned}\Delta g_1 :=& -\sum_{i=1}^{m_1} A_i \int_{t-h_i(t)}^{t} \left[\sum_{j=0}^{p} A_i y_1(s-h_j(s)) + \Delta f + Bu\right] \mathrm{d}s \\ &+ \sum_{i=m_1+1}^{p} A_i y_1(t-h_i(t)) + \Delta f, \\ \Delta g_2 :=& \Delta f + L\Delta C y_1(t) + \sum_{i=m_2+1}^{p} A_i y_1(t-h_i(t)) \\ &- \left\{\sum_{i=1}^{m_2} A_i \int_{t-h_i(t)}^{t} \left[A_0 y_1(s) + \left(\sum_{j=1}^{p} A_j y_1(s-h_j(s))\right)\right.\right. \\ &\left.\left. + \ \Delta f + Bu(s)\right] \mathrm{d}s\right\},\end{aligned}$$

z_t is the segment of $z(s)$ for $t-2H \leqslant s \leqslant t$, $z_t(s) := z(t+s)$ $\forall s \in [-2H, 0]$, and $\|z_t\|_s := \sup_{-2H\leqslant r\leqslant 0}\|z(t+r)\|$. By comparing (6) with (7a)–(7c), it is easy to see that $z(t) = [x^{\mathrm{T}}(t)\ e^{\mathrm{T}}(t)]^{\mathrm{T}}$ $\forall t \geqslant 0$. By the definition of the functions F_1, F_2, and (A1), it can be deduced that

$$\begin{aligned}\|F_1(t,z_t)\| \leqslant& (\|\bar{A}+BK\| + \|BK\|)\cdot\|z(t)\| + \|\Delta g_1\| \\ \leqslant& \ (\|\bar{A}+BK\| + \|BK\|)\cdot\|z(t)\| + \left(\sum_{i=m_1+1}^{p}\|A_i\|\right)\cdot\|z_t\|_s \\ &+ \left(\sum_{i=0}^{p} a_i\right)\cdot\|z_t\|_s + H\left(\sum_{i=1}^{m_1}\|A_i\|\right) \\ &\times \left(\sqrt{2}\|BK\| + \sum_{i=0}^{p}(\|A_i\|+a_i)\right)\cdot\|z_t\|_s, \end{aligned} \tag{8}$$

$$\begin{aligned}\|F_2(t,z_t)\| \leqslant& \|\tilde{A}-LC\|\cdot\|z(t)\| + \|\Delta g_2\| \\ \leqslant& \ \|\tilde{A}-LC\|\cdot\|z(t)\| + \left(\sum_{i=m_2+1}^{p}\|A_i\|\right)\cdot\|z_t\|_s \\ &+ \left(\sum_{i=0}^{p} a_i\right)\cdot\|z_t\|_s + \ c\|L\|\|z(t)\| + H\left(\sum_{i=1}^{m_2}\|A_i\|\right) \\ &\times \left[\ \left(\sum_{i=0}^{p}(\|A_i\|+a_i)\right) + \ \sqrt{2}\|BK\|\right]\cdot\|z_t\|_s, \end{aligned} \tag{9}$$

Clearly, by the definition of F, (8), and (9), it can be deduced that

$$\begin{aligned}\|F(t,z_t)\| &\leqslant \|F_1(t,z_t)\| + \|F_2(t,z_t)\| \\ &\leqslant \left[\|\bar{A}+BK\| + \|BK\| + \|\tilde{A}-LC\| + c\|L\|\right]\cdot\|z(t)\| \\ &\quad + \left[\left(\sum_{i=m_1+1}^{p}\|A_i\|\right) + 2\left(\sum_{i=0}^{p}a_i\right) + \left(\sum_{i=m_2+1}^{p}\|A_i\|\right)\right]\cdot\|z_t\|_s \\ &\quad + H\left(\sum_{i=1}^{m_1}\|A_i\|\right)\left(\sqrt{2}\|BK\| + \sum_{i=0}^{p}(\|A_i\|+a_i)\right)\cdot\|z_t\|_s \\ &\quad + H\left(\sum_{i=1}^{m_2}\|A_i\|\right)\left[\left(\sum_{i=0}^{p}(\|A_i\|+a_i)\right) + \sqrt{2}\|BK\|\right]\cdot\|z_t\|_s.\end{aligned}$$

It follows that the functional $F:\ \mathbb{R}\times C(t_1, 2H, \mathbb{R}^{2n}) \to \mathbb{R}^{2n}$ takes $\mathbb{R}\times$ (bounded sets of $C(t_1, 2H, \mathbb{R}^{2n})$) into bounded sets of $\mathbb{R}^{2n}$. Furthermore, it can be readily obtained that

$$\begin{aligned}&2\|z(t)\|\cdot\|P\|\cdot\|\Delta g_1\| \\ &\quad\leqslant 2\lambda_{\max}(P)\cdot\left[\left(\sum_{i=m_1+1}^{p}\|A_i\|\right) + \left(\sum_{i=0}^{p}a_i\right) + H\left(\sum_{i=1}^{m_1}\|A_i\|\right)\right. \\ &\qquad\left.\times\left(\sqrt{2}\|BK\| + \sum_{i=0}^{p}(\|A_i\|+a_i)\right)\right]\cdot\|z(t)\|\cdot\|z_t\|_s,\end{aligned}\tag{10}$$

$$\begin{aligned}&2\|z(t)\|\cdot\|P\|\cdot\|\Delta g_2\| \\ &\quad\leqslant 2\lambda_{\max}(P)\cdot\left[\left(\sum_{i=m_2+1}^{p}\|A_i\|\right) + \left(\sum_{i=0}^{p}a_i\right) + c\|L\| + H\left(\sum_{i=1}^{m_2}\|A_i\|\right)\right. \\ &\qquad\left.\times\left[\left(\sum_{i=0}^{p}(\|A_i\|+a_i)\right) + \sqrt{2}\|BK\|\right]\right]\cdot\|z(t)\|\cdot\|z_t\|_s.\end{aligned}\tag{11}$$

Let

$$V(z(t)) = z^{\mathrm{T}}(t)Pz(t). \tag{12}$$

The time derivative of $V(z(t))$ along the trajectories of the system (7) is given by

$$\begin{aligned}\dot{V}(z(t)) = z^{\mathrm{T}}(t)&\left(\begin{bmatrix}\bar{A}+BK & -BK \\ 0 & \tilde{A}-LC\end{bmatrix}^{\mathrm{T}}P\right. \\ &\left.+P\begin{bmatrix}\bar{A}+BK & -BK \\ 0 & \tilde{A}-LC\end{bmatrix}\right)z(t) + 2z^{\mathrm{T}}(t)P\begin{bmatrix}\Delta g_1 \\ \Delta g_2\end{bmatrix}\end{aligned}$$

$$
\begin{aligned}
&\leqslant -z^{\mathrm{T}}(t)Qz(t)+2\|z(t)\|\|P\|(\|\Delta g_1\|+\|\Delta g_2\|)\\
&\leqslant -\lambda_{\min}(Q)\|z(t)\|^2+2\|z(t)\|\|P\|\|\Delta g_1\|+2\|z(t)\|\|P\|\|\Delta g_1\|\\
&\leqslant -\lambda_{\min}(Q)\|z(t)\|^2+2\lambda_{\max}(P)\\
&\quad\times\left\{2\left(\sum_{i=0}^{p}a_i\right)+\left(\sum_{i=m_1+1}^{p}\|A_i\|\right)+\left(\sum_{i=m_2+1}^{p}\|A_i\|\right)+c\|L\|\right.\\
&\quad+H\left[\left(\sum_{i=1}^{m_1}\|A_i\|\right)\left(\sqrt{2}\|BK\|+\sum_{i=0}^{p}(\|A_i\|+a_i)\right)+\left(\sum_{i=1}^{m_2}\|A_i\|\right)\right.\\
&\quad\left.\left.\times\left(\sqrt{2}\|BK\|+\left(\sum_{i=0}^{p}(\|A_i\|+a_i)\right)\right)\right]\right\}\|z(t)\|\|z_t\|_s, \qquad (13)
\end{aligned}
$$

in view of (2). Define the decreasing function

$$
\begin{aligned}
g(x)&=\lambda_{\min}(Q)-2(1+x)\lambda_{\max}(P)\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\\
&\quad\times\left\{2\left(\sum_{i=0}^{p}a_i\right)+\left(\sum_{i=m_1+1}^{p}\|A_i\|\right)+\left(\sum_{i=m_2+1}^{p}\|A_i\|\right)+c\|L\|\right.\\
&\quad+H\left[\left(\sum_{i=1}^{m_1}\|A_i\|\right)\left(\sqrt{2}\|BK\|+\sum_{i=0}^{p}(\|A_i\|+a_i)\right)\right.\\
&\quad\left.\left.+\left(\sum_{i=1}^{m_2}\|A_i\|\right)\left(\sqrt{2}\|BK\|+\left(\sum_{i=0}^{p}(\|A_i\|+a_i)\right)\right)\right]\right\},\quad x\geqslant 0.
\end{aligned}
$$

By (iii), we have $g(0)>0$. Consequently, there exists a sufficiently small constant $\varepsilon_1>0$ such that $g(\varepsilon_1)>0$, i.e.,

$$
\begin{aligned}
\varepsilon_2:=g(\varepsilon_1)&=\lambda_{\min}(Q)-2(1+\varepsilon_1)\lambda_{\max}(P)\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\\
&\quad\times\left\{2\left(\sum_{i=0}^{p}a_i\right)+\left(\sum_{i=m_1+1}^{p}\|A_i\|\right)+\left(\sum_{i=m_2+1}^{p}\|A_i\|\right)+c\|L\|\right.\\
&\quad+H\left[\left(\sum_{i=1}^{m_1}\|A_i\|\right)\left(\sqrt{2}\|BK\|+\sum_{i=0}^{p}(\|A_i\|+a_i)\right)\right.
\end{aligned}
$$

$$+\left(\sum_{i=1}^{m_2}\|A_i\|\right)\left(\sqrt{2}\|BK\|+\left(\sum_{i=0}^{p}(\|A_i\|+a_i)\right)\right)\Bigg]\Bigg\} > 0. \tag{14}$$

In the spirit of Theorem 4.2 in Hale [2], with $p(s)=(1+\varepsilon_1)^2 s$, we suppose that

$$V(z(t+r)) < (1+\varepsilon_1)^2 V(z(t)) \quad \forall -2H \leqslant r \leqslant 0,$$

which implies that

$$\|z(t+r)\| < (1+\varepsilon_1)\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\|z(t)\| \quad \forall -2H \leqslant r \leqslant 0. \tag{15}$$

Substituting (15) into (13), it can be shown that

$$\begin{aligned}\dot{V}(y(t)) \leqslant\ & -\lambda_{\min}(Q)\|z(t)\|^2 + 2\lambda_{\max}(P)\,(1+\varepsilon_1)\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\\ &\times\Bigg\{2\left(\sum_{i=0}^{p}a_i\right)+\left(\sum_{i=m_1+1}^{p}\|A_i\|\right)+\left(\sum_{i=m_2+1}^{p}\|A_i\|\right)\\ &\quad + c\|L\| + H\Bigg[\left(\sum_{i=1}^{m_1}\|A_i\|\right)\left(\sqrt{2}\|BK\|+\sum_{i=0}^{p}(\|A_i\|+a_i)\right)\\ &\quad +\left(\sum_{i=1}^{m_2}\|A_i\|\right)\left(\sqrt{2}\|BK\|+\left(\sum_{i=0}^{p}(\|A_i\|+a_i)\right)\right)\Bigg]\Bigg\}\cdot\|z(t)\|^2\\ =\ & -\varepsilon_2\cdot\|z(t)\|^2,\end{aligned} \tag{16}$$

in view of (14). Thus, by Theorem 4.2 in Hale [2] with (12) and (16), we conclude that the system (7) and the system (6) are both globally asymptotically stable. This completes our proof. □

Simply setting $L_1 = L_2 = \emptyset$ in Theorem 2.1, we may obtain the following delay-independent criterion for the global asymptotic stabilizability of system (1).

Corollary 2.1. *The system* (1) *satisfying* (A1) *is globally asymptotically stabilizable by a dynamic observer-based output feedback provided that there exist matrices* $Q > 0$, $K \in \mathbb{R}^{n\times d}$, *and* $L \in \mathbb{R}^{n\times r}$ *such that*

(i) (A_0, B, C) *is jointly stabilizable and detectable*;

(ii) $$\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}\cdot\sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}} > 2\left(\sum_{i=0}^{p}a_i+\sum_{i=0}^{p}\|A_i\|\right)+c\|L\|$$

where $P > 0$ *is the unique solution to the following Lyapunov equation*

$$\begin{bmatrix} A_0 + BK & -BK \\ 0 & A_0 - LC \end{bmatrix}^{\mathrm{T}} P + P \begin{bmatrix} A_0 + BK & -BK \\ 0 & A_0 - LC \end{bmatrix} = -Q.$$

with $A_0 + BK \in H_c$ *and* $A_0 - LC \in H_c$. *In this case, a suitable dynamic observer-based output feedback is given by*

$$u(t) = K\hat{x}(t),$$

$$\dot{\hat{x}}(t) = (A_0 - LC + BK)\hat{x}(t) + Ly(t).$$

Remark 2.1. Note that if $A_i = 0, h_i(t) = 0 \; \forall\, i \in \underline{p}$, $\Delta C(t) = 0$, and $\Delta f = 0$, the assertion of Corollary 2.1 is reduced to the well known fact that a delay-free linear system can be stabilized by a dynamic output feedback provided that (A_0, B, C) is jointly stabilizable and detectable.

Simply setting $L_1 = L_2 = \underline{p}$ in Theorem 2.1, we may obtain the following delay-dependent criterion for the global asymptotic stabilizability of the system (1).

Corollary 2.2. *The system* (1) *satisfying* (A1) *is globally asymptotically stabilizable by a dynamic observer-based output feedback provided that there exist matrices* $Q > 0$, $K \in \mathbb{R}^{n\times d}$, *and* $L \in \mathbb{R}^{n\times r}$ *such that*
(i) $\left(\sum_{i=0}^{p} A_i, B, C\right)$ *is jointly stabilizable and detectable*;
(ii)

$$\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}\sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}} > 2\left(\sum_{i=0}^{p} a_i\right) + c\|L\| + H\cdot\left(\sum_{i=1}^{p}\|A_i\|\right) \times \left(2\sqrt{2}\|BK\| + 2\sum_{i=0}^{p}(\|A_i\| + a_i)\right), \tag{17}$$

where $P > 0$ *is the unique solution to the following Lyapunov equation*

$$\begin{bmatrix} \left(\sum_{i=0}^{p} A_i\right) + BK & -BK \\ 0 & \left(\sum_{i=0}^{p} A_i\right) - LC \end{bmatrix}^{\mathrm{T}} P + P\begin{bmatrix} \left(\sum_{i=0}^{p} A_i\right) + BK & -BK \\ 0 & \left(\sum_{i=0}^{p} A_i\right) - LC \end{bmatrix} = -Q, \tag{18}$$

with $(\sum_{i=0}^{p} A_i) + BK \in H_c$ *and* $(\sum_{i=0}^{p} A_i) - LC \in H_c$. *In this case, a suitable dynamic observer-based output feedback is given by*

$$u(t) = K\hat{x}(t),$$

$$\dot{\hat{x}}(t) = (A_0 - LC + BK)\hat{x}(t) + \sum_{i=1}^{p} A_i\hat{x}(t) + Ly(t).$$

Remark 2.2. It is noted from Corollary 2.2 that even if (A_0, B, C) is not jointly stabilizable and detectable, the system (1) may still be globally asymptotically stabilized by a dynamic observer-based output feedback.

Remark 2.3. By Corollary 2.2, an upper bound of arbitrary time-varying delays without destroying stability is given by $H < \bar{H}$, where

$$\bar{H} = \frac{\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)} \cdot \sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}} - 2\left(\sum_{i=0}^{p} a_i\right) + c\|L\|}{\left(\sum_{i=1}^{p} \|A_i\|\right) \cdot \left(2\sqrt{2}\|BK\| + 2\sum_{i=0}^{p}(\|A_i\| + a_i)\right)}$$

$$\text{if } \left(\sum_{i=1}^{p} \|A_i\|\right) \cdot \left(2\sqrt{2}\|BK\| + 2\sum_{i=0}^{p}(\|A_i\| + a_i)\right) \neq 0,$$

and $\bar{H} = \infty$ otherwise, provided that all conditions of Corollary 2.2 are satisfied.

3. Example

Consider the following uncertain system with time-varying delay described as

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0.1 & 0 \\ 0.1 & -0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0.1 \\ -0.1 & -0.1 \end{bmatrix} x(t - h(t))$$
$$+\Delta f(t, x(t), x(t - h(t))) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \tag{19a}$$

$$y(t) = [-0.5 \;\; 1] x(t) \tag{19b}$$

with

$$\|\Delta f(t, x(t), x(t - h(t)))\| \leqslant 0.01\|x(t)\| + 0.01\|x(t - h(t))\|, \quad 0 \leqslant h(t) \leqslant 0.02.$$

In comparison with (1) and (19), it can be obtained that

$$p = 1, \quad H = 0.02, \quad a_0 = a_1 = 0.01, \quad c = 0,$$

$$A_0 = \begin{bmatrix} 0.1 & 0 \\ 0.1 & -0.1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0.1 \\ -0.1 & 0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [-0.5 \;\; 1].$$

Furthermore it can be readily obtained that $(A_0 + A_1, B, C)$ is jointly stabilizable and detectable. By selecting the parameters

$$K = [-12.1 \;\; -2.1], \quad L = \begin{bmatrix} -17.534 \\ -6.667 \end{bmatrix}, \quad Q = I,$$

it follows that

$$\left(\sum_{i=0}^{1} A_i\right) + BK \in H_c, \quad \left(\sum_{i=0}^{1} A_i\right) - LC \in H_c,$$

$$\lambda_{\max}(P) = 2.1931, \quad \lambda_{\min}(P) = 1.7233,$$

in view of (18). Hence (17) is evidently satisfied, for in this case

$$\begin{aligned}
&\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)} \cdot \sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}} \\
&= 0.2020 > 0.1546 \\
&= 2\left(\sum_{i=0}^{1} a_i\right) + H \cdot \|A_1\| \cdot \left(\sqrt{2}\|BK\| + 2\sum_{i=0}^{1}(\|A_i\| + a_i) + \|LC\|\right).
\end{aligned}$$

Consequently, by Corollary 2.2, we conclude that the system (19) with

$$u(t) = [-12.1 \quad -2.1]\hat{x}(t),$$

$$\dot{\hat{x}}(t) = \begin{bmatrix} -8.667 & 17.634 \\ -15.4335 & 4.567 \end{bmatrix}\hat{x}(t) + \begin{bmatrix} -17.534 \\ -6.667 \end{bmatrix} y(t)$$

is globally asymptotically stable.

With

$$\begin{aligned}
\Delta f(t, x(t), x(t-h(t))) &= \begin{bmatrix} -0.01x_2(t)\sin[x_1(t-h(t)) \cdot x_2(t)] \\ 0.01x_1(t-h(t)) \end{bmatrix}, \\
h(t) &= 0.01 + 0.01\cos(20t),
\end{aligned}$$

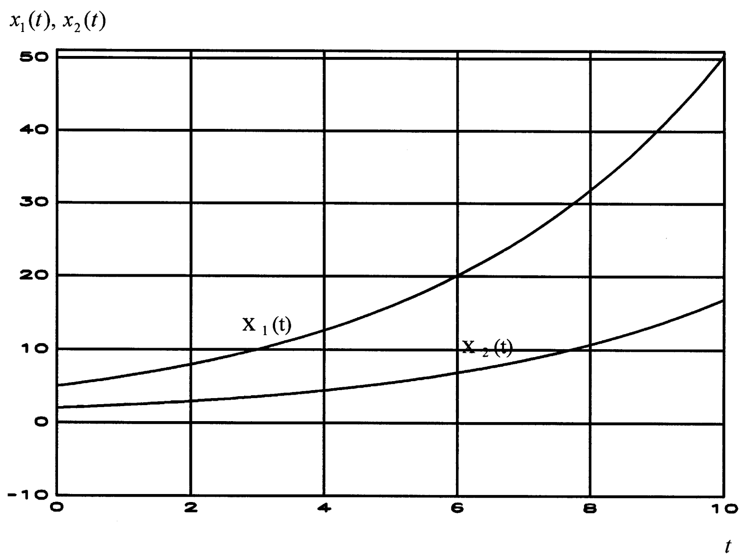

Fig. 1. Typical phase trajectories of the uncontrolled system for (19).

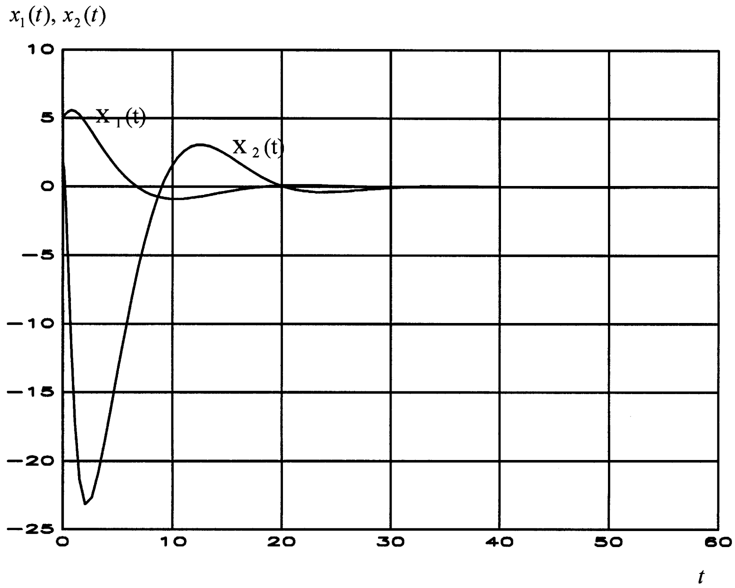

Fig. 2. Typical phase trajectories of the feedback-controlled system for (19).

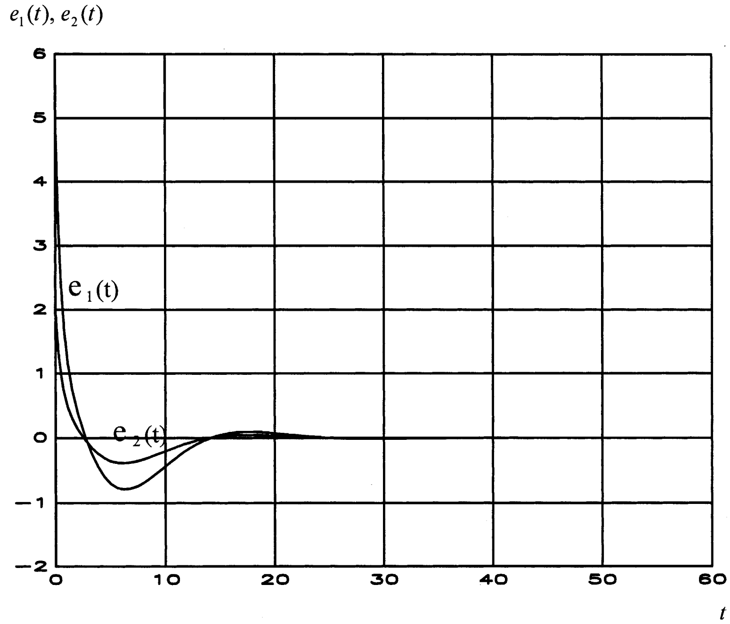

Fig. 3. The errors between states and estimated states ($e_1(t) := x_1(t) - \hat{x}_1(t)$, $e_2(t) := x_2(t) - \hat{x}_2(t)$).

some state trajectories of the (unstable) uncontrolled system are depicted in Fig. 1. In addition, the (stable) state trajectories and the errors between true state trajectories and estimated state trajectories of the feedback-controlled system are depicted in Figs. 2 and 3, respectively. Note that the system (19) is unstable in the case of $u(t) = 0$. In addition, even when (A_0, B, C) is not jointly stabilizable and detectable in this case, the system (19) can still be globally asymptotically stabilized by a dynamic observer-based output feedback. Furthermore, by Corollary 2.2 with Remark 2.3, an upper bound of arbitrary time-varying delay $h(t)$ without destroying stability of system (19) is given by $h(t) < \bar{H} = 0.02826$.

4. Conclusions and discussions

In this paper, the dynamic observer-based output feedback control for a class of uncertain time-delay systems with time-varying delays has been considered. A delay-independent dynamic observer-based output feedback control has been proposed such that the feedback-controlled system with time-varying delays is globally asymptotically stable if some mild conditions are met. An upper bound of arbitrary time-varying delays without destroying stability has been also given such that the asymptotic stability is preserved. The main results may be applicable to a class of nonlinear time-delay system containing linearly bounded uncertainty. However, the dynamic observer-based output feedback control for the nonlinear time-delay system with more general uncertainty other than the form

$$\|\Delta f(t, z_0, z_1, \ldots, z_p)\| \leqslant \sum_{i=0}^{p} a_i \cdot \|z_i\|, \quad \|\Delta C(t)\| \leqslant c.$$

still remains unanswered. This constitutes an interesting future research problem.

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References

[1] J.S. Cheng, J.G. Hsieh, Deterministic control of uncertain feedback systems with time-delay and series nonlinearities, Int. J. Syst. Sci. 26 (1995) 691–701.
[2] J.K. Hale, Theory of Functional Differential Equations, Academic Press, New York, 1977.
[3] F.H. Hsiao, J.G. Hsieh, M.S. Wu, Determination of the tolerable sector of series nonlinearities in uncertain time-delay systems under dynamical output feedback, ASME J. Dyn. Syst. Meas. Control 113 (1991) 525–531.

[4] C. Lin, J.L. Wang, G.-H. Yang, C.B. Soh, Robust controllability and robust closed-loop stability with static output feedback for a class of uncertain descriptor systems, Linear Algebra Appl. 297 (1999) 133–155.
[5] H. Logemann, Destabilizing effects of small time delays on feedback-controlled descriptor systems, Linear Algebra Appl. 272 (1998) 131–153.
[6] M.S. Mahmoud, N.F. Al-Muthairi, Quadratic stabilization of continuous time systems with state-delay and norm-bounded time-varying uncertainties, IEEE Trans. Autom. Control 39 (1994) 2135–2139.
[7] S. Phoojaruenchanachai, K. Furuta, Memoryless stabilization of uncertain linear systems including time-varying state delays, IEEE Trans. Autom. Control 37 (1992) 1022–1026.
[8] J.C. Shen, B.S. Chen, F.C. Kung, Memoryless stabilization of uncertain dynamic delay systems: Riccati equation approach, IEEE Trans. Autom. Control 36 (1991) 638–640.
[9] Y.J. Sun, J.S. Cheng, J.G. Hsieh, C.C. Chen, Feedback control of a class of nonlinear singularly perturbed systems with time delay, Int. J. Syst. Sci. 27 (1996) 589–596.
[10] Y.J. Sun, J.G. Hsieh, Global exponential stabilization for a class of uncertain nonlinear systems with time-varying delay arguments and input dead-zone nonlinearities, J. Franklin Inst. Eng. Appl. Math. 332 (1996) 619–631.
[11] H. Trinh, M. Aldeen, Stabilization of uncertain dynamic delay systems by memoryless feedback controllers, Int. J. Control 59 (1994) 1525–1542.
[12] H.S. Wu, K. Mizukami, Robust stabilization of uncertain linear dynamical systems with time-varying delay, J. Optim. Theory Appl. 82 (1994) 593–606.